BIRATIONAL CONTRACTION OF GENUS TWO TAILS IN THE MODULI SPACE OF GENUS FOUR CURVES. I.

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ABSTRACT. We show that for $\alpha \in (2/3, 7/10)$, the log canonical model $\overline{M}_4(\alpha)$ of the pair $(\overline{M}_4, \alpha\delta)$ is isomorphic to the moduli space $\overline{M}_4^{hs}$ of h-semistable curves constructed in [HH08], and that there is a birational morphism $\Xi : \overline{M}_4^{hs} \to \overline{M}_4(2/3)$ that contracts the locus of curves $C_1 \cup_p C_2$ consisting of genus two curves meeting in a node p such that p is a Weierstrass point of C_1 or C_2. To obtain this morphism, we construct a compact moduli space $\overline{M}_{2,1}^{hs}$ of pointed genus two curves that have nodes, ordinary cusps and tacnodes as singularity, and prove that it is isomorphic to Rulla's flip constructed in his thesis [Rul01].

1. INTRODUCTION

We continue our investigation of the log minimal model program for the moduli space of stable curves. In [HH09], [HH08], [HL10a], [HL10b], Brendan Hassett and the authors have studied the log canonical models

$$\overline{M}_g(\alpha) = \mathrm{Proj} \ \oplus_{n\geq 0} \Gamma(\overline{M}_g, n(K_{\overline{M}_g} + \alpha\delta)), \quad \alpha \in (7/10 - \epsilon, 1]$$

where ϵ is a small positive rational number. Each log canonical model was given a GIT construction and interpreted as a moduli space, and their relations were concretely described. For $g \geq 3$, the first divisorial contraction T occurs at $\alpha = 9/11$ followed by a flip at $\alpha = 7/10$.

$$
\begin{array}{c}
\overline{M}_g(1) \simeq \overline{M}_g \\
\downarrow {\scriptstyle T} \\
\overline{M}_g(\tfrac{9}{11}) \simeq \overline{M}_g^{ps} \qquad\qquad\qquad\qquad \overline{M}_g(\tfrac{7}{10} - \epsilon) \simeq \overline{M}_g^{hs} \\
{\scriptstyle \Psi} \searrow \qquad\qquad \swarrow {\scriptstyle \Psi^+} \\
\overline{M}_g(\tfrac{7}{10}) \simeq \overline{M}_g^{cs}
\end{array}
$$

1

Here $\overline{M}_g^{ps}$, $\overline{M}_g^{cs}$ and $\overline{M}_g^{hs}$ denote the moduli space of pseudostable curve [Sch91], of c-semistable curves and of h-semistable curves respectively [HH08]. In this paper, we shall prove that for $g = 4$, the sequence of maps above is followed by another small contraction:

Theorem 1.   (1) $\overline{M}_4(\alpha) \simeq \overline{M}_4^{hs}$ *for $2/3 < \alpha < 7/10$.*
   (2) *There is a birational morphism $\Xi : \overline{M}_4^{hs} \to \overline{M}_4(2/3)$ contracting the locus of Weierstrass genus two tails.*

A *genus two tail* of a curve C is a subcurve of genus two that meets the rest of the curve in one node. It is called a *Weierstrass genus two tail* if moreover the attaching node is fixed under the hyperelliptic involution. Abusing terminology, we will call C itself a (Weierstrass) genus two tail if it has a (Weierstrass) genus two tail as a subcurve. Our proof uses the results from Hosung Kim's thesis [Kim08] and thus applies only to the case $g = 4$, but we believe that the statement is true for any $g \geq 4$. In fact, we may remove use of [Kim08] by constructing the GIT quotient space of *6-Hilbert semistable curves*: Examination of the slope of the linearization suggests that $\overline{M}_g(2/3)$ is isomorphic to the GIT quotient of the Hilbert scheme of the sixth Hilbert points of bicanonical curves. More precisely, $K_{\overline{\mathcal{M}}_g} + \frac{2}{3}\delta$, when pulled back to the Hilbert scheme of ν-canonical curves, is proportional to the linearization [HH08, Equation (5.3)] when $\nu = 2$ and $m = 6$. But we have not been able to construct the GIT quotient mainly because of our lack of understanding of finite Hilbert stability of curves. Recently, Ian Morrison and Dave Swinarski made a breakthrough [MS09], constructing examples of small genera curves that are 6-Hilbert stable. Building upon this work, we plan to carry out the GIT and give modular interpretations of $\overline{M}_g(2/3)$ and the small contraction Ξ for any $g \geq 4$.

To prove our main theorem, we also construct a new moduli space $\overline{M}_{2,1}^{hs}$ of pointed curves of genus two that allows nodes, ordinary cusps and tacnodes as singularities, and prove that it is isomorphic to the flip of $\overline{M}_{2,1}$ given in [Rul01, Section 3.9][1]. It is constructed by using GIT, and the method can be generalized to give new compactifications of $M_{g,n}$ for any $g \geq 2$ and n. Details will be given in our forthcoming work [HLS].

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[1]Rigorously what we want to say here is 'the flip of $\overline{M}_{2,1}^{ps}$', but it will be clear from the context what we mean and there is no need to introduce a new moduli space.

2. H-STABLE CURVES AND POINTED CURVES

In [HH08], we classified bicanonically embedded curves of genus $g \geq 4$ whose mth Hilbert point is GIT semistable for $m \gg 0$. Such a curve C is said to be *h-semistable*, and is characterized by the following properties:

(1) C has nodes, ordinary cusps and tacnodes as singularity;
(2) a smooth rational component of C meets the rest of the curve in ≥ 3 points counting multiplicity;
(3) a genus one subcurve of C meets the rest of the curve in ≥ 3 points counting multiplicity;
(4) no *elliptic chain* is admitted.

By [HH08, Proposition 8.7, Corollary 8.8], every h-semistable curve of genus four has finite automorphisms. Indeed, h-semistable curves with genus one subcurve are listed in Figure 1 and none of them admits a *weak elliptic chain*. When $g = 4$, the elliptic chain condition is redundant, and we will not use elliptic chains or weak elliptic chains in the present article but point the interested readers to [HH08, Definition 2.4].

FIGURE 1. h-semistable curves of genus four with genus one subcurve

It essentially follows from this that the moduli stack $\overline{\mathcal{M}}_4^{\mathrm{hs}}$ of h-stable curves of genus four is Deligne-Mumford and the coarse moduli space $\overline{M}_4^{\mathrm{hs}}$ has only finite quotient singularities. We let δ^{hs} denote the divisor of singular curves in the moduli stack. It has two irreducible components $\delta_{\mathrm{irr}}^{\mathrm{hs}}$ consisting of irreducible curves and δ_2^{hs} consisting of genus two tails. Since the morphism $\overline{\mathcal{M}}_4^{\mathrm{hs}} \to \overline{M}_4^{\mathrm{hs}}$ is unramified (as opposed to $\overline{\mathcal{M}}_g \to \overline{M}_g$ which is ramified along δ_1), we shall abuse the notation and use them to denote the corresponding divisor classes in the moduli space. In [HH08, Theorem 9.1], it is shown that the linearization on the Hilbert scheme used in the GIT quotient construction of $\overline{M}_4^{\mathrm{hs}}$ descends to (a positive rational multiple of)

$$K_{\overline{M}_4^{\mathrm{hs}}} + (7/10 - \epsilon)\delta^{\mathrm{hs}}$$

on $\overline{M}_4^{\mathrm{hs}}$, where ϵ is a small positive rational number. Thus $K_{\overline{M}_4^{\mathrm{hs}}} + (7/10 - \epsilon)\delta^{\mathrm{hs}}$ is an ample $\mathbb{Q}$-divisor on $\overline{M}_4^{\mathrm{hs}}$. Due to [HH08, Lemma 2.8 and Equation (9.1)], we have

Lemma 1. *There is a natural isomorphism*

$$\overline{M}_4(2/3) \simeq \mathrm{Proj} \oplus_{m \geq 0} \Gamma\left(\overline{M}_4^{\mathrm{hs}}, m(K_{\overline{M}_4^{\mathrm{hs}}} + \frac{2}{3}\delta^{\mathrm{hs}})\right).$$

We shall therefore study the map associated to $K_{\overline{M}_4^{hs}} + 2/3\delta^{hs}$. An overarching principle of the program is that the intermediate maps contract the boundary divisors $\Delta_1, \Delta_2, \ldots$ one by one, so we seek extremal rays for $K_{\overline{M}_4^{hs}} + \frac{2}{3}\delta^{hs}$ generated by a curve in the boundary divisor δ_2^{hs}.

Via the degree two finite map

$$(1) \qquad\qquad j : \overline{M}_{2,1} \times \overline{M}_{2,1} \to \Delta_2 \hookrightarrow \overline{M}_4$$

we can pass to $\overline{M}_{2,1}$ whose birational geometry has been extensively studied in [Rul01].[2] But to use his results in studying the birational geometry of $\overline{M}_4^{hs}$, we need to have a map for $\overline{M}_4^{hs}$ corresponding to j in (1) above, which breaks δ_2^{hs} into two components: any curve in δ_2^{hs} is by definition a union of two genus two curves meeting in one node. Since such component curves may have an ordinary cusp or a tacnode, we need to construct a compact moduli space of pointed curves allowing these singularities. This is the task we carry out in §4. Here, we sketch the construction and summarize the main properties of the new moduli space.

Let $C_1 \cup_p C_2$ be a curve in δ_2^{hs}, where C_i are of genus two. Restricting the polarization to C_i yields $\omega_{C_i}^{\otimes 2}(2p)$, which leads us to consider the GIT of pointed genus two curves (C, p) embedded in $\mathbb{P}^4$ by the complete linear system $|\omega_C^{\otimes 2}(2p)|$. We say that (C, p) is ν-*canonically embedded* if C is embedded in $\mathbb{P}^N$ by $|\omega_C^{\otimes \nu}(\nu p)|$, $N+1 = (2\nu-1)(g-1)+\nu$. In our case $\nu = 2$, (C, p) is said to be *biconically embedded*. We recall the basic GIT setup from [Swi08a]. Let $d = 6$, $N + 1 = 5$, and $P(m) = 6m - 1$. Let Hilb be the Hilbert scheme parametrizing the subschemes of $\mathbb{P}^N$ whose Hilbert polynomial is $P(m)$. Let J denote the subscheme of Hilb $\times \mathbb{P}^N$ consisting of the points associated to bicanonical pointed curves (C, p) such that C is smooth and $p \in C$ is a smooth point.[3] The parameter space we use is $\overline{J}$, the closure of J inside Hilb $\times \mathbb{P}^N$, linearized by

$$(2) \qquad\qquad \mathcal{L}_m := 3p_1^*(\textstyle\bigwedge \pi_*\mathcal{O}_{\mathcal{C}}(m)) + 2m^2 p_2^*\mathcal{O}_{\mathbb{P}^N}(1)$$

where $\pi : \mathcal{C} \to$ Hilb is the universal curve. Here, the balancing factor $2m^2/3$ was chosen so that the GIT quotient space parametrizes exactly the pointed curves that we desire. Such a balancing factor was employed in [BS08] and [Swi08a], and for ν-canonical pointed curves, it turns out that $\nu m^2/(2\nu - 1)$ works.[4] We say that (C, p) is *Hilbert (semi)stable* if the corresponding point in $\overline{J}$ is GIT (semi)stable with respect to the standard SL_{N+1} action and $\mathcal{L}_m$ for all $m \gg 0$.

Theorem 2. *A bicanonical pointed curve* (C, p) *of genus two is Hilbert stable if it is h-stable i.e. it satisfies the following properties:*

    (1) C *has nodes, ordinary cusps and tacnodes as singularity;*

    (2) *A rational component of* C *has* ≥ 3 *special points counting multiplicity;*

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[2]This approach is originally due to Brendan Hassett.

[3]Swinarski's J is actually bigger than our J but this does not affect the outcome as their closure are equal.

[4] We learned of this from Dave Swinarski, and gratefully acknowledge his help.

(3) C *does not have a subcurve of genus one.*
(4) p *is a smooth point.*

Since these have finite automorphisms, there is no strictly semistable point and the quotient space $\overline{M}_{2,1}^{hs}$ has finite quotient singularities. Consequently the corresponding moduli stack is Deligne-Mumford. As in the case of $\overline{\mathcal{M}}_4$, we have the morphism

$$\overline{\mathcal{M}}_{2,1}^{hs} \times \overline{\mathcal{M}}_{2,1}^{hs} \to \overline{\mathcal{M}}_4^{hs}$$

of stacks given by attaching two h-stable pointed curves at the marked points forming a node. This leads to a straightforward generalization of [GKM02, Lemma 1.1]:

Lemma 2. *An effective curve C in the product $\overline{M}_{2,1}^{hs} \times \overline{M}_{2,1}^{hs}$ is numerically equivalent to the sum of curves $C_1 \times \{pt\} + \{pt\} \times C_2$ where C_i is obtained by projecting C to the ith factor. In particular, if a line bundle on the product is nef on each slice $\overline{M}_{2,1}^{hs} \times pt$ and $pt \times \overline{M}_{2,1}^{hs}$ of the product, then it is nef.*

The linearization on $\overline{J}$ descends to an ample line bundle on the quotient space $\overline{J}/\!\!/SL$, which can be written in terms of tautological classes on the quotient space [Swi08b]: Let $\pi : \mathcal{C} \to \overline{J}$ be the universal curve and ψ denote the bundle of cotangent lines. In the polarization formula below, the first term $\det \pi_* \omega_{\mathcal{C}}(m)$ comes from the curve part and the second, the point part. We follow [Mum77, Page 106] and write $\det \pi_* \omega_{\mathcal{C}}(m)$ in terms of the Hodge class λ, the boundary divisor δ_{irr} of irreducible curves and ψ. Using Swinarski's polarization formula [Swi08b] with $\nu = 2$, $n = 1$ (number of marked points), $m_1' = \frac{2}{3}m^2$ (balancing factor), $A_1 = \nu a_1 = 2$ (weight), we get

$$\det \pi_* \omega_{\mathcal{C}}(m) + \frac{2}{3}m^2 Q = \binom{m}{2}\widetilde{\kappa}_1 + \lambda + (\binom{m}{2} + m)\psi + (m(6m-1) + \frac{2}{3}m^2)Q$$

where $Q = -\frac{1}{5}(\widetilde{\kappa} + \lambda + \psi)$. Using the relations

$$\delta_1 = 5\lambda - \frac{1}{2}\delta_{irr}$$
$$\widetilde{\kappa} = 12\lambda - \delta = 7\lambda - \frac{1}{2}\delta_{irr}$$

we may rewrite $Q = -\frac{1}{5}(8\lambda + \psi - \frac{1}{2}\delta_{irr})$ and obtain

$$\text{(♣)} \quad \begin{aligned} &\binom{2m}{2}(7\lambda - \frac{1}{2}\delta_{irr}) + \lambda + \binom{2m}{2}\psi - \frac{1}{5}(m(6m-1) + \frac{2}{3}m^2)(8\lambda + \psi - \frac{1}{2}\delta_{irr}) \\ &= (\frac{10}{3}m^2 - \frac{27}{5}m + 1)\lambda + (\frac{2}{3}m^2 - \frac{4}{5}m)\psi - (\frac{1}{3}m^2 - \frac{2}{5}m)\delta_{irr} \\ &\sim (10 - \epsilon(m))\lambda + 2\psi - \delta_{irr} \end{aligned}$$

where $\epsilon(m) = \frac{21m-150}{100m^2 - 120m}$ tends to 0 as $m \to \infty$.

3. Construction of the birational morphism Ξ

We shall first prove that $|\ell(K_{\overline{M}_4^{hs}} + 2/3\delta^{hs})|$ is basepoint free for sufficiently large and divisible ℓ. We shall accomplish this by showing that

(i) $|\ell(K_{\overline{M}_4^{hs}} + 2/3\,\delta^{hs})|$ gives rise to an isomorphism on $\overline{M}_4^{hs} \setminus \delta_2^{hs} \cup \overline{H}_4^{hs}$;
(ii) $K_{\overline{M}_4^{hs}} + 2/3\,\delta^{hs}$ is nef on δ_2^{hs};
(iii) $K_{\overline{M}_4^{hs}} + 2/3\,\delta^{hs}$ is nef on $\overline{H}_4^{hs}$.

Since (i) ∼ (iii) together imply that $K_{\overline{M}_4^{hs}} + 2/3\,\delta^{hs}$ is nef, the basepoint freeness follows by Kawamata-Viehweg basepoint freeness theorem. That $K_{\overline{M}_4^{hs}} + 2/3\delta^{hs}$ is nef on $\overline{H}_4^{hs}$ is shown in [HL10b]: It gives rise to a projective birational morphism with exceptional locus $\widetilde{B}_5$. Proofs of (i) and (ii) will be given in subsequent sections. Then we shall put everything together and establish Theorem 1 in § 3.3.

3.1. $K_{\overline{M}_4^{hs}} + 2/3\,\delta^{hs}$ **gives rise to an isomorphism over** $\overline{M}_4^{hs} \setminus \delta_2^{hs} \cup \overline{H}_4^{hs}$. Let ϵ be a small positive rational number. Note that we can write

$$(3) \qquad K_{\overline{M}_4^{hs}} + 2/3\,\delta^{hs} = a\left(K_{\overline{M}_4^{hs}} + (7/10 - \epsilon)\,\delta^{hs}\right) + b\left(K_{\overline{M}_4^{hs}} + 5/9\,\delta^{hs}\right)$$

with positive coefficients $a = 10/(13 - 90\epsilon)$ and $b = (3 - 90\epsilon)/(13 - 90\epsilon)$. Since $K_{\overline{M}_4^{hs}} + (7/10 - \epsilon)\,\delta^{hs}$ is ample, it suffices to show that $K_{\overline{M}_4^{hs}} + 5/9\,\delta^{hs}$ is semiample over $\overline{M}_4^{hs} \setminus \delta_2^{hs} \cup \overline{H}_4^{hs}$. For this end, we exploit the moduli space $\mathrm{Chow}_{4,1}/\!\!/\mathrm{SL}_4$ of Chow semistable canonical curves of genus four. This space was carefully studied by H. Kim [Kim08] in which she proves among other things that:

Theorem 3. [Kim08] *A complete connected curve of genus four has Chow semistable (resp. stable) canonical image if it is h-stable and not contained in $\delta_2^{hs} \cup \overline{H}_4^{hs}$ (resp. $\delta_2^{hs} \cup \overline{H}_4^{hs} \cup T_1^3$).*

Here, T_1^3 denotes the locus of *3-pointed elliptic tails* i.e. h-stable curves with a genus one subcurve meeting the rest in 3 points counting multiplicity.

Corollary 1. *We have a birational map*

$$\Theta : \overline{M}_4^{hs} \dashrightarrow \mathrm{Chow}_{4,1}/\!\!/\mathrm{SL}_4$$

which is a regular morphism on $\overline{M}_4^{hs} \setminus (\delta_2^{hs})^o$*, where* $(\delta_2^{hs})^o := \delta_2^{hs} \setminus (\delta_2^{hs} \cap \overline{H}_4^{hs})$.

Proof. Since $\overline{M}_4^{hs} \setminus \delta_2^{hs} \cup \overline{H}_4^{hs} \cup T_1^3$ and the corresponding locus in $\mathrm{Chow}_{4,1}/\!\!/\mathrm{SL}_4$ are both geometric quotients, they are coarse moduli spaces for the moduli functor

$$\mathcal{M}'(S) = \left\{ f : C \to S \;\middle|\; \begin{array}{l} \text{(i)} \quad \text{f is flat;} \\ \text{(ii)} \quad \text{geometric fibre } C_s \text{ is h-stable of genus four,} \\ \text{(iii)} \quad \text{has no 3-pointed elliptic tails, and} \\ \text{(iv)} \quad \text{has no genus two subcurves.} \end{array} \right\}.$$

Hence there is a natural isomorphism φ between them. We claim that it can be extended to a regular morphism on T_1^3. By [Kim08, Theorem 3.20], curves in T_1^3 are identified in $\mathrm{Chow}_{4,1}/\!\!/\mathrm{SL}_4$ and represented by a single point $p^\star$. Let $\sigma : B = \mathrm{Spec}\,k[[t]] \to \overline{M}_4^{hs}$ be a map from a smooth curve whose generic point misses $\delta_2^{hs} \cup \overline{H}_4^{hs} \cup T_1^3$. After a base change, we obtain an h-stable family $\pi : C \to B$ that corresponds to σ. Since this family satisfies Kim's Chow semistability criterion, after a re-embedding $C \hookrightarrow \mathbb{P}(\pi_*\omega_\pi) \simeq \mathbb{P}_B^3$, we obtain a map $B \to \mathrm{Chow}_{4,1}$, and in turn a map to the quotient space $\mathrm{Chow}_{4,1}/\!\!/\mathrm{SL}_4$. Necessarily the special point of B maps to $p^\star$. This means that φ

is compatible with the constant map collapsing T_1^3 to $p^\star$, and φ extends across T_1^3 by Hartogs' theorem.

Likewise, Θ is regular on $\overline{H}_4^{hs}$. In fact, it simply maps $\overline{H}_4^{hs}$ to a point, since the Chow cycle of any smooth hyperelliptic curve of genus four is twice that of the twisted cubic. $\qquad\square$

Lemma 3. $\Theta^* \mathcal{O}(+1)$ *is a positive rational multiple of*

$$K_{\overline{M}_4^{hs}} + 5/9 \delta^{hs} - c \delta_2^{hs}$$

for some $c > 0$.

Proof. Since $\overline{M}_4^{hs}$ and $\mathrm{Chow}_{4,1}/\!\!/SL_4$ are good quotients of normal varieties, they are normal. So Θ extends to a regular morphism on $\overline{M}_4^{hs}$ away from a codimension two locus, and $\Theta^* \mathcal{O}(+1)$ is well defined. By [Mum77, Theorem 5.15], the polarization on $\mathrm{Chow}_{4,1}/\!\!/SL_4$ pulls back to a positive rational multiple of $9\lambda - \delta$ on $\mathrm{Chow}_{4,1}$. So, away from δ_2^{hs}, the strict transform of $\mathcal{O}(+1)$ on $\overline{M}_4^{hs}$ is $9\lambda - \delta_0^{hs}$ and we have the discrepancy equation

$$(4) \quad \Theta^* \mathcal{O}(+1) = 9\lambda - \delta_0^{hs} - c' \delta_2^{hs} = (9\lambda - \delta^{hs}) - c\delta_2^{hs} = \frac{9}{13}(K_{\overline{M}_4^{hs}} + 5/9 \delta^{hs}) - c\delta_2^{hs}.$$

We shall prove that there is a curve $C \subset \delta_2^{hs}$ such that

(1) Θ is regular along C;
(2) $(9\lambda - \delta^{hs}).C < 0$;
(3) $\delta_2^{hs}.C < 0$.

Then dotting both sides of (4) with C gives $c > 0$.

Let (C_1, p) be a one-parameter family of pointed genus two curves obtained by taking C_1 as the genus two curve corresponding to the vital curve $\{1, 1, 2, 2\}$ and p, a ramification point of C_1. Let C be the curve in δ_2^{hs} obtained by gluing (C_1, p) and a fixed smooth curve C_2 of genus two at a fixed Weierstrass point on C_2. Since C is contained in the hyperelliptic locus, Θ is regular along C by Corollary 1. Note that C completely misses the flipping locus of elliptic bridges, so $(9\lambda - \delta^{hs}).C = (9\lambda - \delta^{ps}).C$ where δ^{ps} is the divisor of singular curves in $\overline{M}_4^{ps}$. Employing the discrepancy formula, we find that $(9\lambda - \delta^{ps}).C$ is a positive rational multiple of $(K_{\overline{M}_4} + \frac{5}{9}\delta - \frac{26}{9}\delta_1).C$. Here we slightly abused notation and let C denote the corresponding curve in $\overline{M}_4$, which is reasonable since it does not meet the exceptional locus. To compute the intersection on $\overline{M}_4$, since C_2 is fixed, we may replace it by a suitable reducible rational curve and pass to the vital curve V associated to $\{1, 2, 2, 5\}$ [KM96], and compute instead the intersection with V (Figure 2). This is easily done by using [KM96, Theorem 1.3.(6)]: $K_{\overline{M}_4} + \alpha\delta + (11\alpha - 9)\delta_1$ pulls back to

$$\sum_{s=1}^{\lfloor \frac{g+1}{2} \rfloor} \left\{ \frac{13}{4g+2}s(g+1-s) + 2(\alpha-2) \right\} \widetilde{B}_{2s} + \sum_{s=1}^{\lfloor \frac{g}{2} \rfloor} \left\{ \frac{13}{4g+2}s(g-s) + \frac{\alpha-2}{2} \right\} \widetilde{B}_{2s+1} + (11\alpha - 9)\widetilde{B}_3$$

of which intersection with V is $\frac{19}{2}(\alpha - \frac{19}{12})$. It follows that $(9\lambda - \delta^{hs}).C < 0$. Similarly, one can easily show that $\delta_2^{hs}.C < 0$.
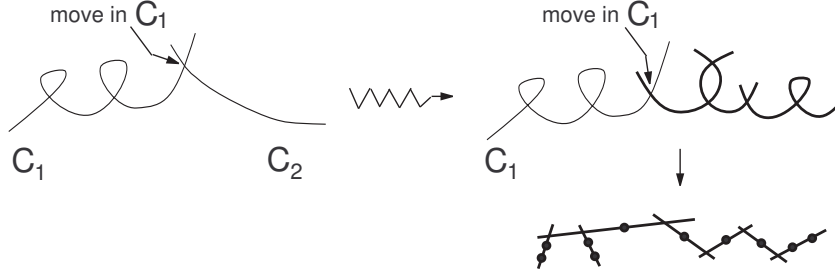
FIGURE 2. Passing to the vital curve $\{1, 2, 2, 5\}$

$\square$

It follows from the lemma that for any h-stable curve D not in $\delta_2^{hs} \cup \overline{H}_4^{hs}$, there is a section of (a sufficiently large and divisible multiple of) $K_{\overline{M}_4^{hs}} + 5/9\delta^{hs}$ that does not vanish at the point representing D. Since $K_{\overline{M}_4^{hs}} + (7/10 - \epsilon)\delta^{hs}$ is ample, it follows that the linear combination (3) gives rise to an isomorphism on $\overline{M}_4^{hs} \setminus \delta_2^{hs} \cup \overline{H}_4^{hs}$.

3.2. $K_{\overline{M}_4^{hs}} + 2/3\delta^{hs}$ **is nef on** δ_2^{hs}. Fix a stable 1-pointed curve (C_o, p_o) of genus two and let $j' : \overline{M}_{2,1}^{hs} \to \overline{M}_4^{hs}$ denote the map sending (C, p) to $C \cup_{p=p_o} C_o$, the h-stable curve obtained by gluing C and C_o such that p and p_o are identified to a node. If $j : \overline{M}_{2,1} \to \overline{M}_4$ is the corresponding map, then the commutative diagram of maps

$$
\begin{array}{ccc}
\overline{M}_{2,1} & \xrightarrow{\ j\ } & \overline{M}_4 \\
\big\downarrow & & \big\downarrow \\
\overline{M}_{2,1}^{hs} & \xrightarrow{\ j'\ } & \overline{M}_4^{hs}
\end{array}
$$

gives rise to a commutative diagram of the Picard groups. Due to Lemma 2, to establish the nefness of $K_{\overline{M}_4^{hs}} + 2/3\delta^{hs}$ on δ_2^{hs}, it suffices to show that its pullback to $\overline{M}_{2,1}^{hs}$ by j' is nef.

By the discrepancy formula [HH09, Lemma 4.1], $K_{\overline{M}_4^{hs}} + \alpha\delta^{hs}$ corresponds to $K_{\overline{M}_4} + (\alpha - 2)\delta + (11\alpha - 9)\delta_1$, which pulls back to

$$13\lambda + (\alpha - 2)(\delta_{irr} - \psi) + (12\alpha - 11)\delta_{1,\{1\}}.$$

For $\alpha = 2/3$, this is a positive rational multiple of

$$-\delta_{irr} - 12\delta_{1,\{1\}} + 40\psi$$

which is in turn proportional to ([Rul01, Proposition 3.3.7]) $D = 3(-\delta_{irr} - 12\delta_{1,\{1\}} + 40\psi)$. Rulla showed that D gives rise to a rational map $\varphi_{\overline{Q}}$ on $\overline{M}_{2,1}$ which contracts

$\Delta_{1,\{1\}}$ and

$$\overline{W}_2^1 = \{(C,p)\,|\,p \text{ is a Weierstrass point}\}$$

and that $\varphi_{\overline{Q}}$ becomes regular after a flip $\overline{M}_{2,1} \dashrightarrow Z$. Then by the contraction theorem, there is a birational morphism $Z \to Y$ contracting $\Delta_{1,\{1\}}$, and D remains nef on Y [Rul01, Section 3.9].

Proposition 1. $\overline{M}_{2,1}^{\mathrm{hs}}$ *is isomorphic to Rulla's flip* Y.

Proof. Note that $\overline{M}_{2,1}^{\mathrm{hs}}$ and Y are isomorphic away from a locus of codimension two since both admit birational maps from $\overline{M}_{2,1}$ contracting $\Delta_{1,\{1\}}$. It remains to show that an ample line bundle on $\overline{M}_{2,1}^{\mathrm{hs}}$ is ample on Y.

We start with $K_{\overline{\mathcal{M}}_4} + \alpha\delta$ on $\overline{M}_4$. We shall prove that

$$j^*(K_{\overline{\mathcal{M}}_4} + \alpha\delta) = j^*(13\lambda + (\alpha-2)\delta) = 13\lambda + (\alpha-2)(\delta_{\mathrm{irr}} + \delta_{1,\{1\}}) + (2-\alpha)\psi$$

is ample on both Y and on $\overline{M}_{2,1}^{\mathrm{hs}} = \mathrm{Hilb}/\!\!/\mathrm{SL}_5$ (Theorem 4) when $\alpha = 7/10 - \epsilon$ for some small positive rational number ϵ. Note that for $\alpha = 7/10 - \epsilon$, it is proportional to

$$(5) \qquad -\epsilon'\delta_{\mathrm{irr}} + (13-\epsilon')\delta_{1,\{1\}} + (13+\epsilon')\psi, \quad \epsilon' = 10\epsilon.$$

On Y, $\delta_{1,\{1\}}$ is contracted and (5) can be written as a positive rational linear combination of $C \sim \delta_{\mathrm{irr}} - 18\delta_{1,\{1\}} + 45\psi$ and $D \sim -\delta_{\mathrm{irr}} - 12\delta_{1,\{1\}} + 40\psi$. Thus it is ample on Y since C and D are extremal on the two dimensional nef cone of Y [Rul01, § 3.9].

In §2, Equation (♣), we found the linearization $\mathcal{L}_m$ on the GIT quotient $\mathrm{Hilb}/\!\!/\mathrm{SL}_5$ to be a positive rational multiple of $(10 - \epsilon(m))\lambda - (\delta_{\mathrm{irr}} - 2\psi)$ where $\epsilon(m) = \frac{21m-150}{100m^2-120m}$. On the other hand,

$$K_{\overline{\mathcal{M}}_4} + (7/10 - \epsilon)\delta \sim (10-\epsilon)\lambda - (\delta_{\mathrm{irr}} + \delta_1 + \delta_2)$$

pulls back under j to

$$(10-\epsilon)\lambda - \delta_{\mathrm{irr}} - \delta_{1,\{1\}} + \psi = (10-\epsilon)\lambda - (5\lambda - \tfrac{1}{2}\delta_{\mathrm{irr}}) + \psi \sim (10-\epsilon')\lambda - \delta_{\mathrm{irr}} + 2\psi$$

which agrees with the polarization on $\mathrm{Hilb}/\!\!/\mathrm{SL}_5$ for $m \gg 0$. $\qquad\square$

In view of the isomorphism $Y \simeq \overline{M}_{2,1}^{\mathrm{hs}}$, the discussion preceding Proposition 1 implies that the pull back of $K_{\overline{M}_4^{\mathrm{hs}}} + \tfrac{2}{3}\delta^{\mathrm{hs}}$ by j' is nef, giving rise to a birational morphism contracting the proper transform of $\overline{W}_2^1$.

3.3. **Proof of Theorem 1.** Recall from §3.1 that the positive rational linear combination $K_{\overline{M}_4^{\mathrm{hs}}} + \alpha\,\delta^{\mathrm{hs}}$ can be written as

$$a(K_{\overline{M}_4^{\mathrm{hs}}} + (7/10 - \epsilon)\delta^{\mathrm{hs}}) + b(K_{\overline{M}_4^{\mathrm{hs}}} + 2/3\,\delta^{\mathrm{hs}})$$

for some positive rational numbers a and b, and a small positive rational number ϵ. We have shown that the first factor is ample and the second is semiample. It follows that the sum is ample and Theorem 1.(1) is established.

We shall now consider the exceptional locus of the birational morphism associated to $|\ell(K_{\overline{M}_4^{hs}} + 2/3\,\delta^{hs})|$ for sufficiently large and divisible ℓ. We have seen in §3.1 that the morphism is an isomorphism away from $\delta_2^{hs} \cup \overline{H}_4^{hs}$. On $\overline{H}_4^{hs}$, $K_{\overline{M}_4^{hs}} + 2/3\,\delta^{hs}$ contracts $\widetilde{B}_5 = \delta_2^{hs} \cap \overline{H}_4^{hs}$ [HL10b]. Therefore, the exceptional locus is completely contained in δ_2^{hs}.

On the other hand, on $\overline{M}_{2,1}^{hs}$, we noted in the previous section that $K_{\overline{M}_4^{hs}} + 2/3\,\delta^{hs}$ gives rise to a birational morphism with exceptional locus the proper transform in $\overline{M}_{2,1}^{hs}$ of $\overline{W}_2^1$. Recall the maps $\overline{M}_{2,1}^{hs} \xrightarrow{j'} \overline{M}_{2,1}^{hs} \times \overline{M}_{2,1}^{hs} \to \delta_2^{hs}$. By Lemma 2, a curve in $\overline{M}_{2,1}^{hs} \times \overline{M}_{2,1}^{hs}$ is numerically equivalent to a sum of curves from each factor. Hence on δ_2^{hs}, the exceptional locus of the morphism associated to $|\ell(K_{\overline{M}_4^{hs}} + 2/3\,\delta^{hs})|$ is the locus of Weierstrass genus two tails.

Remark 1. Since $K_{\overline{M}_4^{hs}} + 2/3\,\delta^{hs}$ pulls back to D which is extremal, it follows that $K_{\overline{M}_4^{hs}} + \alpha\,\delta^{hs}$ is not nef on $\overline{M}_4^{hs}$ for $\alpha < 2/3$. Also, it is shown in [HL10b] that $K_{\overline{M}_4^{hs}} + 2/3\,\delta^{hs}$ restricts to an extremal divisor on $\overline{H}_4^{hs}$.

4. Moduli space of h-stable pointed curves of genus two

In this section, we work out the GIT of bicanonically embedded pointed curves discussed in §2. We retain the notations $d, N, P(m), \mathrm{Hilb}, \overline{J}$ and $\mathcal{L}_m$ from §2. GIT of $\overline{J}$ gives rise to the Hilbert semistable bicanonical pointed curves. Alternatively, one can employ the Chow variety in place of Hilb. Let Chow denote the Chow variety of curves of degree d in $\mathbb{P}^N$. Given a bicanonical model of a pointed curve (C, p), we call the corresponding point in $\mathrm{Chow} \times \mathbb{P}^N$ the *Chow point* of (C, p) and denote it by $\mathrm{Ch}(C, p)$. Then we define $J' \subset \mathrm{Chow} \times \mathbb{P}^N$ in a similar manner and consider the GIT of $\overline{J}'$, linearized by (the restriction to $\overline{J}'$ of)

$$\mathcal{O}_{\mathrm{Chow}}(3) \boxtimes \mathcal{O}_{\mathbb{P}^N}(4)$$

which pulls back by the Chow cycle map to the leading coefficient of the balanced linearization $\mathcal{L}_m$ (§2, Equation (2)) on the Hilbert scheme (see §4.1 below). We say that (C, p) is *Chow (semi)stable* if its Chow point is GIT (semi)stable. Swinarski completely works out the theory for pointed curves $(C, p_1, \ldots, p_n)$ embedded by $(\omega_C(\sum p_i))^\nu$ for $\nu \geq 5$, and proves that such a curve if Hilbert stable if and only if it is Deligne-Mumford stable.[5] This result may also be derived from [BS08]. We shall work out the case $g = 2$, $n = 1$ and $\nu = 2$.

Definition 1. A pointed curve (C, p) of genus two is said to be *c-semistable* if

(1) C has nodes, ordinary cusps and tacnodes as singularity;
(2) A rational component of C has ≥ 3 special points counting multiplicity;

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[5]Swinarski actually considers the problem in more general setting of Hassett's weighted pointed curve.

(3) A genus one subcurve meets the rest of the curve in ≥ 2 points without counting multiplicity;
(4) p is simple.

It is *c-stable* if it is c-semistable and C has no tacnodes and no subcurve of genus one.

Definition 2. A pointed curve (C, p) of genus two is said to be *h-stable* if

(1) C has nodes, ordinary cusps and tacnodes as singularity;
(2) A rational component of C has ≥ 3 special points counting multiplicity;
(3) C does not have a subcurve of genus one.
(4) p is simple.

Theorem 4. *A bicanonical pointed curve* (C, p) *of genus two is Hilbert stable if and only if it is h-stable. There is no strictly semistable point.*

Therefore, the quotient space $\overline{M}_{2,1}^{hs} := \overline{J} /\!\!/ SL_5$ has only finite quotient singularities, and the corresponding moduli stack $\overline{\mathcal{M}}_{2,1}^{hs}$ is Deligne-Mumford.

Remark 1. Although $\overline{M}_{2,1}^{hs}$ is a coarse moduli space for a well defined separated moduli functor, it is not a *stable modular compactification* in the sense of Smyth [Smy09]. A stable modular compactification in general does not allow any curve with singularity worse than ordinary node.

Theorem 5. *A bicanonical pointed curve* (C, p) *of genus two is Chow (semi)stable if and only if it is c-(semi)stable.*

4.1. Unstable pointed curves. Let $\mathcal{L}'_\ell$ denote

$$p_1^* \mathcal{O}_{Chow}(+1) \otimes p_2^* \mathcal{O}_{\mathbb{P}^N}(\ell)$$

on $Chow \times \mathbb{P}^N$. Due to a result of Mumford, Fogarty and Knudsen, we have

$$(6) \qquad \bigwedge \pi_* \mathcal{O}_\mathcal{C}(m) = \binom{m}{2} Ch^* \mathcal{O}_{Chow}(+1) + \text{lower degree terms}$$

Hence $\mathcal{L}'_{4/3}$ pulls back via $Ch \times 1_{\mathbb{P}^N}$ to a positive rational multiple of the top coefficient of the linearization $3p_1^*(\bigwedge \pi_* \mathcal{O}_\mathcal{C}(m)) + 2m^2 p_2^* \mathcal{O}_{\mathbb{P}^N}(1)$, where $Ch : Hilb \to Chow$ is the Chow cycle map. It follows that

Lemma 4. (C, p) *is Hilbert unstable if its Chow point is unstable with respect to* $\mathcal{L}'_{4/3}$.

We shall employ this lemma extensively to deduce the geometric invariant theoretic property of Hilbert points from the GIT of corresponding Chow points, and vice versa.

Remark 2. We shall interchangeably use 1-PS ρ of GL_5 with integral weights $(r_0, \ldots, r_4)$ and its corresponding 1-PS of SL_5 with rational weights $(r_0 - \frac{1}{5} \sum r_i, \ldots, r_4 - \frac{1}{5} \sum r_i)$. Also, note that

$$\mu^{\mathcal{L}'_{4/3}}(Ch(C, p), \rho) = \mu^{\mathcal{O}_{Chow}(+1)}(Ch(C), \rho) + \frac{4}{3} \mu^{\mathcal{O}_{\mathbb{P}^N}(+1)}(p, \rho).$$

Lemma 5. (C, p) *is Chow unstable if* p *is not a smooth point.*

Proof. Choose coordinates so that $p = [1, 0, \dots, 0]$ and let ρ be a 1-PS with weights $(1, 0, \dots, 0)$. Let $\nu : \widetilde{C} \to C$ be the normalization and let $p', p'' \in \widetilde{C}$ be the points $(p' = p''$ if p is a cusp) over p. Then by [Sch91, Lemma 1.4], we have

$$e_\rho(C) \geq e_\rho(\widetilde{C})_{p'} + e_\rho(\widetilde{C})_{p''} \geq 2 \cdot 1^2$$

and hence

$$\mu(Ch(C), \rho) \leq -2 + \frac{2 \cdot 6}{5} = \frac{2}{5}.$$

The maximum of the negative of the weights at p is $-(1 - 1/5) = -4/5$. Hence

$$\mu(Ch(C, p), \rho) \leq \frac{2}{5} + \frac{4}{3} \cdot (-\frac{4}{5}) = -\frac{2}{3}.$$

$\square$

Lemma 6. *If C has a triple point, then (C, p) is Chow unstable.*

Proof. We follow the proof of [Mum77, Proposition 3.1]. Choose coordinates so that $[1, 0, \dots, 0]$ is a triple point, and let ρ be the 1-PS of GL_{N+1} with weights $(1, 0, \dots, 0)$. Then $\mu(Ch(C), \rho) \leq -3 + \frac{2d}{N+1}$ and $\mu(p, \rho) \leq \frac{1}{N+1}$. Hence the Hilbert-Mumford index of the Chow point of (C, p) satisfies

$$\mu(Ch(C, p), \rho) \leq -3 + \frac{2d}{N+1} + \frac{4}{3} \frac{1}{N+1} = -3 + \frac{8}{3} < 0.$$

$\square$

Note that a genus two curve cannot have a singularity of the form $y^2 = x^m$ for $m \geq 6$, for any such singularity would contribute ≥ 3 to the arithmetic genus.

Lemma 7. *If C has a ramphoid cusp i.e. a singularity of the form $y^2 = x^5$, then (C, p) is Chow unstable.*

Proof. Let $p \in C$ be a ramphoid cusp, $\widetilde{C} \to C$ be the normalization and $q, r \in \widetilde{C}$, the points over p. Note that the analysis in [HH08, Lemma 7.2] does not depend on the particular embedding but only on the degree. Applying the proof to our situation, we obtain a 1-PS ρ with weights $(5, 3, 1, 0, 0)$ with respect to suitable coordinates, so that

$$\mu(Ch(C), \rho) \leq -25 + \frac{2d}{N+1}(5 + 3 + 1) = -25 + \frac{108}{5} = -\frac{17}{5}.$$

The maximum of the negative of the weights is $\frac{9}{5}$. Hence

$$\mu(Ch(C, p), \rho) \leq -\frac{17}{5} + \frac{4}{3} \cdot \frac{9}{5} = -1.$$

$\square$

Lemma 8. *If C has a multiple component, (C, p) is Chow unstable.*

Proof. In [HH08, Lemma 7.4], it is shown that there is a 1-PS ρ with weights $(3, 2, 1, 0, 0)$ such that

$$\mu(\mathrm{Ch}(C), \rho) \leq -18 + \frac{2d}{N+1}(3 + 2 + 1)$$

which in our case is equal to $-18 + \frac{12}{5}6 = -\frac{18}{5}$. The maximum of the negative of the weights is $\frac{6}{5}$, and we obtain

$$\mu(\mathrm{Ch}(C, p), \rho) \leq -\frac{18}{5} + \frac{4}{3}\frac{6}{5} = -2.$$

$\square$

Definition 3. An *elliptic bridge* is a pointed curve of the form $(C := E \cup_{q_0, q_1} R, p)$ where E is an elliptic curve, R is a rational curve meeting E in two nodes q_0 and q_1, and the marked point p is a simple point of R.

Proposition 2. *Let* $(C^\star = E \cup_{q_0, q_1} R, p)$ *be a bicanonical elliptic bridge such that* $E = R_0 \cup_y R_1$ *consists of two rational curves meeting in one tacnode* y. *Then there is a 1-PS* ρ *with respect to which the* m*th Hilbert point has the Hilbert-Mumford index*

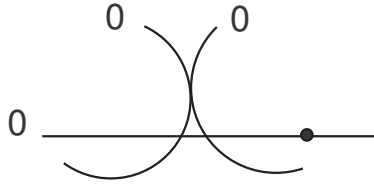$$\mu^{\mathcal{L}_m}((C^\star, p), \rho) = 2m^2 - 7m + 5 + \frac{2}{3}m^2(-3) = -7m + 5.$$

FIGURE 3.

Proof. Assume $q_i \in R_i$. We have

$$\omega_{C^\star}|R_i = \omega_{R_i}(2y + q_i) \simeq \mathcal{O}_{R_i}(+1).$$

Hence $\omega_{C^\star}^{\otimes 2}(2p)$ restricts to $\mathcal{O}_{R_i}(+2)$ on R_i. On R, it restricts to $\mathcal{O}_R(-4+2q_0+2q_1+2p) \simeq \mathcal{O}_R(+2)$. Hence $C^\star$ can be parametrized by:

$$\begin{array}{rcl}
[s_0, t_0] & \mapsto & [\ s_0^2, \quad s_0 t_0, \quad t_0^2, \quad 0, \quad 0] \\
[s_1, t_1] & \mapsto & [\ 0, \quad s_1 t_1, \quad s_1^2, \quad t_1^2, \quad 0] \\
[s, t] & \mapsto & [\ s^2, \quad 0, \quad 0, \quad t^2, \quad st]
\end{array}$$

From this parametrization, we obtain the ideal $I_{C^\star}$ of $C^\star$ embedded in $\mathbb{P}^4$ by $\omega_{C^\star}^{\otimes 2}(2p)$. Let $\rho : \mathbb{C}^* \to SL_5$ with weights $(3, -2, -7, 3, 3)$ which comes from automorphisms of $C^\star$. Using the Gröbner basis algorithm of [HHL10], we obtain the filtered Hilbert function of $C^\star$ with respect to ρ as

$$\mu([C]_m, \rho) = 2m^2 - 7m + 5.$$

On the other hand, the contribution from the marked point is -3. With respect to $\mathcal{L}_m$ with balancing factor $2m^2/3$, the Hilbert-Mumford index is

$$(7) \qquad \mu^{\mathcal{L}_m}((C^\star, p), \rho) = 2m^2 - 7m + 5 + \frac{2}{3}m^2(-3) = -7m + 5.$$

$\square$

In particular, $(C^\star, p)$ is Hilbert unstable with respect to ρ.

Corollary 2. *Bicanonical elliptic bridges are Hilbert unstable.*

Proof. Consider the basin of attraction of the bicanonical elliptic bridge $C^\star = R_0 \cup R_1 \cup R$ with respect to ρ, where $C^\star$ and ρ are as in Proposition 2. The local versal deformation space of a tacnode is given by $y^2 = x^4 + ax^2 + bx + c$ where x parametrizes the tangent space of (the branches at) the tacnode and y is a local parameter of the tacnode (at a branch). The tangent space of (the branches at) the tacnode $[0, 0, 1, 0, 0]$ is parametrized by x_1/x_2, on which ρ acts with weight $-2 - (-7) = +5$. Hence the ρ action on the local versal deformation space of the tacnode has weights $(+10, +15, +20)$. On the other hand, at the node $q_0 = [1, 0, 0, 0, 0]$, ρ acts on the two tangent directions x_1/x_0 and x_4/x_0 with weights -5 and 0 respectively. Hence it acts on the local versal deformation space of q_0 with weight -5. It follows that the basin of attraction with respect to ρ contains an arbitrary smoothing of the tacnode but no smoothing of the nodes. $\square$

Corollary 3.   (1) *Elliptic bridges and the corresponding tacnodal pointed curves are Chow strictly semistable with respect to $\rho^{\pm 1}$;*
   (2) *The basin of attraction of the Chow point of $(C^\star, p)$ with respect to ρ (resp. ρ^{-1}) contain all elliptic bridges (resp. c-semistable pointed curves with a tacnode).*

Note that this completely classifies c-semistable pointed curves that are attracted to $(C^\star, p)$ since any 1-PS coming from the automorphism group of $(C^\star, p)$ is an integral power of ρ.

Proof. (6) and (7) imply that $\mu^{\mathcal{L}'_{4/3}}((C, p), \rho^{\pm 1}) = 0$. We have already analyzed the basin of attraction with respect to ρ in Corollary 2. Since ρ^{-1} acts on the deformation spaces with the weights of opposite signs, we conclude that the basin with respect to ρ^{-1} contain all smoothings of the node and no smoothing of the tacnode. $\square$

Lemma 9. *Let $C^* = E \cup_t R$ be a bicanonical curve consisting of a rational cuspidal curve E and a smooth rational curve R meeting in a tacnode t. Then for any point $p \in R$, (C^*, p) is Chow unstable.*

Proof. Restricting $\omega_C^{\otimes 2}(2p)$ we get

$$\omega_C^{\otimes 2}(2p)|E \simeq \mathcal{O}_E(4t), \quad \omega_C^{\otimes 2}(2p)|R \simeq \mathcal{O}_R(-4)(2p + 4t) \simeq \mathcal{O}_R(2).$$

So E and R can be parametrized by

$$[s, t] \mapsto [s^4, s^2 t^2, st^3, t^4, 0]$$

and
$$[u, v] \mapsto [0, 0, uv, u^2, v^2].$$
The cusp is at $q = [1, 0, \ldots, 0]$ and the tacnode, at $t = [0, 0, 0, 1, 0]$. Let ρ be the 1-PS with weight $(0, 2, 3, 4, 2)$. Using the parametrization, we easily find the ideal of C^* and the Hilbert-Mumford index $\mu([C]_m, \rho)$ by employing the Gröbner basis algorithm of [HHL10]. We have
$$\mu([C]_m, \rho) = -4m^2 + 4m$$
and
$$\mu(\mathrm{Ch}(C), \rho) = \lim_{m \to \infty} \frac{1}{m^2} \mu([C]_m, \rho) = -4.$$
The maximum of the negative of the weights is ≤ 11 and we find that for any $p \in R$,
$$\mu(\mathrm{Ch}(C^*, p), \rho) \leq -4 + \frac{4}{3}\frac{11}{5} = -\frac{16}{15} < 0.$$
$\square$

Lemma 10. *If* C *has a genus one subcurve meeting the rest in one tacnode,* (C, p) *is Chow unstable for any* $p \in C$.

Proof. This follows since C is in the basin of attraction of C^* from Lemma 9: Consider the cusp q whose local equation is $(x_2/x_0)^2 = (x_1/x_0)^3$. Its local versal deformation space is defined by $(x_2/x_0)^2 = (x_1/x_0)^3 + a(x_1/x_0) + b$, so $\mathbb{G}_m$ acts on it via ρ with weight $(+4, +6)$. Thus the basin of attraction $A_\rho([C]_m)$ (and $A_\rho(\mathrm{Ch}(C))$) contains arbitrary smoothing of the cusp. Consequently, $\mathrm{Ch}(C, p)$ is in the basin of attraction of $\mathrm{Ch}(C^*, p^*)$ where $p^* = \lim_{\alpha \to 0} \rho(\alpha).p$. If p is not on the smooth rational component, then (C, p) is Chow unstable by Propositions 3 and 4. So we assume that p is on the smooth rational component of C, which forces p^* to be on the smooth rational component R of C^*. In this case we have
$$\mu(\mathrm{Ch}((C, p)), \rho) = \mu(\mathrm{Ch}((C^*, p^*)), \rho) < 0$$
by Lemma 9.
$\square$

4.2. **Proof of semistability.** It all starts with the following crucial theorem due to David Swinarski:

Theorem 6. [Swi08a] *A bicanonically embedded* n*-pointed curve* $(C, p_1, \ldots, p_n)$ *is Hilbert stable (and hence Chow stable) if* C *is smooth and* p_i*'s are distinct smooth points of* C.

To get things rolling, we also need a pointed version of [Mum77, Proposition 5.5]:

Proposition 3. *Let* $C \subset \mathbb{P}^5$ *be a genus two curve of degree six and* $p \in C$ *be a point such that* (C, p) *is Chow semistable. Then for any subcurve* C_1 *of* C, *we have*
$$|\deg C_1 - 2\deg_{C_1}(\omega_C(p))| \leq \frac{w}{2}$$
where $w = \#(C_1 \cap \overline{C - C_1})$.

Proof. Following Mumford, let $L_1 = \{x_{n_1+1} = \cdots = x_5 = 0\}$ be the smallest linear subspace containing C_1, and let ρ denote the 1-PS defined by

$$\rho(t).x_i = \begin{cases} t\,x_i, & i \le n_1 \\ x_i, & i > n_1. \end{cases}$$

By the Chow semistability of (C, p), we have

$$-e_\rho(C) + \frac{12}{5}(n_1 + 1) + \frac{4}{15}(n_1 + 1) - \frac{4}{3}\delta(p) \ge 0$$

where $\delta(p) = 0$ if $p \notin L_1$ and $\delta(p) = 1$ if $p \in L_1$. Passing to the partial normalization $C_1 \coprod \overline{C - C_1}$, we have

$$e_\rho(C) \ge w + 2 \deg C_1.$$

Combining the two inequalities above, we obtain

(8) $$2 \deg C_1 + w \le \frac{12}{5}(n_1 + 1) + \frac{4}{15}(n_1 + 1) - \frac{4}{3}\delta(p).$$

Since a subcurve of C is of genus zero or one, it is embedded by a non-special linear system. Thus $n_1 + 1 = \deg C_1 + 1 - g_1$ and substituting it in (8) yields

$$-\frac{w}{2} \le \deg C_1 - 4g_1 + 4 - 2w - 2\delta(p) = \deg C_1 - 2\deg_{C_1}(\omega_C(p)).$$

Applying this inequality to $C_2 = \overline{C - C_1}$ and using

$$0 = \deg_C - 2 \deg \omega_C(p) = \sum_{i=1}^{2} \deg C_i - 2 \deg_{C_i}(\omega_C(p))$$

we obtain $\deg C_1 - 2 \deg_{C_1}(\omega_C(p)) \le \frac{w}{2}$. $\qquad\square$

Proposition 4. *Let $(\mathcal{C}, \Sigma) \to \operatorname{Spec} k[[t]]$ be a family of Chow semistable pointed curves of genus g whose generic fibre $\mathcal{C}_\eta$ is smooth. Here $\Sigma : \operatorname{Spec} k[[t]] \to \mathcal{C}$ is necessarily a section of smooth points. If $\Phi : \mathcal{C} \to \mathbb{P}^{3g-2}_{k[[t]]}$ is an embedding such that $\Phi^*_\eta(\mathcal{O}(1)) = \omega^{\otimes 2}_{\mathcal{C}_\eta/k((t))}(2\Sigma|\mathcal{C}_\eta)$, then $\mathcal{O}(1) = \omega^{\otimes 2}_{\mathcal{C}/k[[t]]}(2\Sigma)$.*

In view of the relation between the Hilbert semistability and the Chow semistability, we can safely replace Chow by Hilbert in the statement.

Proof. The proof is due to Mumford and the assertion essentially follows from Proposition 3. We just point out that the extra data of section do not effect the argument. Let C_i denote the components of $\mathcal{C}_0$. Then $\mathcal{O}(1) \simeq \left(\omega^{\otimes 2}_{\mathcal{C}/k[[t]]}(2\Sigma)\right)(\sum r_i C_i)$ for some r_i which we may assume to be nonnegative integers such that $\min\{r_i\} = 0$. Let $C_+ = \cup_{r_i > 0} C_i$ and $C_o = \cup_{r_i = 0} C_i$. Then

$$\#(C_+ \cap C_o) \le \deg_{C_+} \mathcal{O}_\mathcal{C}(\sum r_i C_i) = \deg C_+ - 2 \deg_{C_+} \omega_{\mathcal{C}_0}(p).$$

which contradicts Proposition 3 unless $r_i = 0$ for all i. $\qquad\square$

The proposition implies that if (C, p) is Chow semistable bicanonical pointed curve of genus two, then

(1) C is nondegenerate and;
(2) a rational component of C has three special points counting multiplicity and;
(3) C has no elliptic tail.

For instance, suppose that C had an elliptic tail. Then $C = E_1 \cup_q E_2$ where E_i are subcurves of genus one meeting each other in one node q, and the bicanonical system $\omega_C^{\otimes 2}(2p)$ is not very ample on E_i if $p \notin E_i$.

We now complete the proof of Theorems 4 and 5. Let (C, p) be a c-semistable bicanonical pointed curve. Consider a smoothing $\pi : (\mathcal{C}, \Sigma) \to \operatorname{Spec} k[[t]]$ of C and embed in $\mathbb{P}_{k[[t]]}^{3g-2}$ by choosing a frame for $\pi_*(\omega_{\mathcal{C}/k[[t]]}^{\otimes 2}(2\Sigma))$. This induces a map $\operatorname{Spec} k((t)) \to (\operatorname{Chow} \times \mathbb{P}^N)^{ss}$. Applying the semistable replacement theorem, we obtain a map $\operatorname{Spec} k[[t]] \to (\operatorname{Chow} \times \mathbb{P}^N)^{ss}$ and the corresponding Chow semistable family $(\mathcal{D}, \Sigma') \to \operatorname{Spec} k[[t]]$ whose generic member is isomorphic to that of $(\mathcal{C}, \Sigma)$. It remains to prove that they agree at the special fibre. Note that $\mathcal{D}_0$ and C necessarily have the same Deligne-Mumford stabilization. So $\mathcal{D}_0 = C$ is evident if C has no tacnode and no elliptic bridge as there is no other c-semistable curve that has the same Deligne-Mumford stabilization. Consider the elliptic bridge $(C^\star, p)$ from Proposition 2. The only c-semistable pointed curves that have the same Deligne-Mumford stabilization as $(C^\star, p)$ are elliptic bridges and the tacnodal ones. But these are contained in the basin of attractions of $(C^\star, p)$, and we conclude that all elliptic bridges and c-semistable pointed curves with a tacnode are Chow semistable [HH08, Lemma 4.3]. It also follows that if C is c-stable then it is Chow stable since otherwise C would be contained in a basin of attraction of $(C^\star, p)$, the unique c-semistable pointed curve with infinite automorphisms.

Note that an h-stable pointed curve without a tacnode is c-stable by definition and hence Chow stable. By Lemma 4, it is Hilbert stable. Let (C, p) be an h-stable curve with a tacnode. Suppose it is Hilbert unstable. Then there is a 1-PS ρ' such that $\mu^{\mathcal{L}_m}((C, p), \rho') < 0$ for $m \gg 0$. Let C^* denote the flat limit $\lim_{t \to 0} \rho'(t).C$. Since (C, p) is c-semistable, $\mu^{\mathcal{L}'_{3/4}}((C, p), \rho') = 0$, which implies that C^* is also c-semistable. But a c-semistable pointed curve with infinite automorphisms is of the form $(C^\star, p)$, and it follows from Corollary 3 that ρ' is a positive multiple of ρ^{-1}. But this is a contradiction to Proposition 2 which implies

$$\mu^{\mathcal{L}_m}((C, p), \rho^{-1}) = 7m - 5 > 0.$$

REFERENCES

[BS08]   Elizabeth Baldwin and David Swinarski. A geometric invariant theory construction of moduli spaces of stable maps. *Int. Math. Res. Pap. IMRP*, (1):Art. ID rpn 004, 104, 2008.

[GKM02] Angela Gibney, Sean Keel, and Ian Morrison. Towards the ample cone of $\overline{M}_{g,n}$. *J. Amer. Math. Soc.*, 15(2):273–294 (electronic), 2002.

[HH08]   Brendan Hassett and Donghoon Hyeon. Log minimal model program for the moduli space of curves: the first flip, 2008. arXiv:0806.3444v1 [math.AG].

[HH09]   Brendan Hassett and Donghoon Hyeon. Log canonical models for the moduli space of curves: the first divisorial contraction. *Trans. Amer. Math. Soc.*, 361(8):4471–4489, 2009.

[HHL10] Brendan Hassett, Donghoon Hyeon, and Yongnam Lee. Stability computation via Gröbner basis. *Journal of Korean Mathematical Society*, 47(1):41–62, 2010.

[HL10a] Donghoon Hyeon and Yongnam Lee. Log minimal model program for the moduli space of stable curves of genus three. *Mathematical Research Letters*, 2010. to appear.

[HL10b] Donghoon Hyeon and Yongnam Lee. A new look at the moduli space of hyperelliptic curves. *Mathematische Zeitschrift*, 264(2):317–326, 2010.

[HLS]   Donghoon Hyeon, Yongnam Lee, and David Swinarski. New compactifications of moduli space of stable pointed curves. in preparation.

[Kim08] Hosung Kim. Chow stability of curve of genus 4 in $\mathbb{P}^3$, 2008. arXiv:0806.0731v3 [math.AG].

[KM96]   Sean Keel and James McKernan. Contractible extremal rays on $\overline{M}_{0,n}$, 1996. arXiv.org/alg-geom/9607009.

[MS09]   Ian Morrison and David Swinarski. Groebner techniques for low degree hilbert stability, 2009. arXiv:0910.2047v1 [math.AG].

[Mum77] D. Mumford. Stability of projective varieties. *Enseignement Math. (2)*, 23(1-2):39–110, 1977.

[Rul01] William Rulla. The birational geometry of $\overline{M}_3$ and $\overline{M}_{2,1}$, 2001. Ph. D. thesis, University of Texas at Austin.

[Sch91] David Schubert. A new compactification of the moduli space of curves. *Compositio Math.*, 78(3):297–313, 1991.

[Smy09] D. Smyth. Towards a classification of modular compactifications of $\mathcal{M}_{g,n}$, 2009. arXiv:0902.3690v1[math.AG].

[Swi08a] D. Swinarski. Geometric invariant theory and moduli spaces of pointed curves. *Thesis, Columbia University,* 2008.

[Swi08b] D. Swinarski. private communication, 2008.

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